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# Bipartite Riemann–Finsler geometry and Lorentz violation

V. Alan Kostelecký [a,*], N. Russell [b], R. Tso [c]

[a] Physics Department, Indiana University, Bloomington, IN 47405, USA
[b] Physics Department, Northern Michigan University, Marquette, MI 49855, USA
[c] Physics Department, Embry-Riddle Aeronautical University, Prescott, AZ 86301, USA

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ABSTRACT

Bipartite Riemann–Finsler geometries with complementary Finsler structures are constructed. Calculable examples are presented based on a bilinear-form coefficient for explicit Lorentz violation.



A famous example of Riemann–Finsler geometry is Randers geometry [1], which involves a Riemann metric enhanced by a 1-form. Its popularity stems partly from its simplicity and calculability, with relatively compact expressions attainable for many geometric quantities (see, e.g., Ref. [2]). It also has multiple links to physical situations. Perhaps the simplest example involves a relativistic charged massive particle minimally coupled to a background electromagnetic 1-form potential in $(3+1)$-dimensional spacetime, for which the possible motions lie along the geodesics of a pseudo-Randers metric.

A large class of Riemann–Finsler geometries, including Randers geometry, has recently been linked to Lorentz and CPT violation in realistic effective field theory [3]. The basic idea is that motions of classical particles in the general realistic effective field theory for Lorentz and CPT violation in curved spacetime, the Standard-Model Extension (SME) [4], follow geodesics in pseudo-Riemann–Finsler spacetimes from which corresponding Riemann–Finsler geometries can be constructed. The Lorentz and CPT violation could arise in a fundamental theory unifying quantum physics and gravity such as strings [5], with the SME describing the resulting effects at attainable energies [6,7]. These notions about Riemann–Finsler geometries have application in a variety of related contexts [8–21].

Among the novel geometries are Riemann–Finsler spaces of simplicity and calculability comparable to the Randers case. One surprise is the existence of another calculable Riemann–Finsler space, termed $b$ space, which is also determined by a 1-form and has Finsler structure complementary to that of Randers space. Physically, the corresponding pseudo-Riemann–Finsler geometry is associated with the geodesic motion of a fermion in the presence of chiral CPT-odd Lorentz violation in $(3+1)$-dimensional pseudo-Riemann spacetime [13].

In this Letter, we explore the existence of other complementary pairs of Finsler structures in this class of geometries. For bipartite Finsler structures constructed from the Riemann metric $r_{jk}$ and a nonnegative symmetric bilinear form $s_{jk}$, $j,k=1,2,3,\ldots,n$, we show that when $s_{jk}$ has a single positive eigenvalue the corresponding Riemann–Finsler geometry has a natural complement. Some properties of these bipartite spaces are derived, including the connection between $r$-parallel and Berwald spaces. As explicit examples, we examine special cases of $H$ spaces that have complementary bipartite structures of this type. In $(3+1)$-dimensional spacetime, the corresponding pseudo-Riemann–Finsler structure governs the geodesic motion of a fermion in the presence of CPT-even Lorentz violation. We also identify isomorphisms between Randers space, $b$ space, $H$ space, and $H^\perp$ space. The notation and conventions adopted below are those of Ref. [3].

A bipartite structure is a particular function on the tangent bundle $TM$ of the background spacetime manifold $M$. In terms of $n$-dimensional positions $x^j$ and velocities $y^j$, this function $F(x,y)$ takes the form [3]

$$F(x,y)=\rho+\sigma,\quad \rho:=\sqrt{y^j r_{jk} y^k},\ \sigma:=\pm\sqrt{y^j s_{jk} y^k},\tag{1}$$

where either sign of $\sigma$ can be chosen. Both $r_{jk}$ and $s_{jk}$ are generically functions of $x^j$, and indeed in the corresponding pseudo-Riemann–Finsler geometries a position dependence of the SME coefficients is natural in a gravitational background [4,22–25]. Note that using the inverse Riemann metric $r^{jk}$ to raise an index on $r_{jk}(x)$ and $s_{jk}(x)$ produces linear operators $r^j{}_k(x)\equiv\delta^j{}_k$ (the Kronecker delta) and $s^j{}_k(x)$, respectively.

The bipartite structure $F$ is positive for the positive sign of $\sigma$ and is positive for the negative sign of $\sigma$ when the nonzero eigenvalues of $s^j{}_k$ are less than one, corresponding to the assumption of perturbative Lorentz violation. Also, $F$ is positive homogeneous in $y^j$ of order one. Moreover, $F$ is $C^\infty$ regular on the slit tangent bundle $TM\backslash S$, where $S=S_0\cup S_1$ includes the usual slit

* Corresponding author.
*E-mail address:* kostelec@indiana.edu (V. Alan Kostelecký).

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$S_0 = \{y\colon y^j = 0\}$ and the slit extension $S_1 = \{y\colon s^j{}_k y^k = 0,\ y^j \neq 0\}$. Typically, $F$ is $y$ local, but for certain choices of $s_{jk}$ the slit extension $S_1$ is empty and $F(x, y)$ becomes $y$ global.

With the above conditions, the bipartite structure $F$ becomes a Finsler structure if it has strong convexity, which occurs when the corresponding Finsler metric $g_{jk}$ is positive definite on $TM\backslash S$. This metric is readily calculated to be

$$g_{jk} = \frac{F}{\rho} r_{jk} - \rho\sigma\kappa_j\kappa_k + \frac{F}{\sigma} s_{jk}, \tag{2}$$

where $\kappa_j := \rho_{y^j}/\rho - \sigma_{y^j}/\sigma$. We show below that for the cases of interest here $g_{jk}$ is indeed positive definite on $TM\backslash S$. A more general result establishing conditions on $s_{jk}$ sufficient for strong convexity of $F$ would be of interest.

For the bipartite Finsler structure, the Cartan torsion is found to take the simple form

$$C_{jkl} = -\frac{1}{2}\rho\sigma \sum_{(jkl)} \kappa_j \kappa_{kl}, \tag{3}$$

where the sum spans cyclic permutations of $j$, $k$, $l$. Here, $\kappa_{jk} := \rho_{y^j y^k}/\rho - \sigma_{y^j y^k}/\sigma$ involves the second $y^j$ derivatives of $\rho$ and $\sigma$. Since the Cartan torsion is nonzero whenever $\sigma$ is nontrivial, the Deicke theorem [26] implies the bipartite structure is then noneuclidean as a Minkowski norm. With nonzero $\sigma$, the bipartite geometry therefore cannot be a Riemann geometry.

Our interest in this work lies in special bipartite geometries that appear in complementary pairs. To investigate this explicitly, in what follows we restrict $s^j{}_k(x)$ to have rank $m$ with one nonzero positive eigenvalue $\varsigma(x)$ of multiplicity $m$, where $\varsigma < 1$ for $F$ to be positive on $TM\backslash S$. It follows that $s^j{}_k = \varsigma \hat{s}^j{}_k$, where $\hat{s}^j{}_k$ is idempotent, $\hat{s}^2 = \hat{s}$. Note that in an appropriate basis $\hat{s}^j{}_k$ is the diagonal matrix with $m$ unit entries and $n - m$ zero entries, $\hat{s} = I_m$. We thus have

$$s^2 = \varsigma s, \quad 0 < \varsigma < 1. \tag{4}$$

Note also that if $m = 0$ then $s_{jk} = 0$ and the geometry is Riemann, while if $m = n$ then $s_{jk} = \varsigma r_{jk}$ and the geometry is again Riemann but with a metric scaled by $(1 \pm \sqrt{\varsigma})^2$.

To show strong convexity of $F$ for $s_{jk}$ satisfying the condition (4), which amounts to showing positive definiteness of the Finsler metric (2) in this limit, consider the determinant of $g_{jk}$. Some calculation reveals it can be written as

$$\det(g_{jk}) = \left(\frac{F}{\rho}\right)^{n+1} \left(\frac{S}{\sigma}\right)^{m-1} \det(r_{jk}), \tag{5}$$

where the function $S := \varsigma\rho + \sigma$ generalizes the function $B$ of Ref. [3] and is always nonzero for $y^j \neq 0$, with its sign matching the sign of $\sigma$. The standard argument [2] for positive definiteness of $g_{jk}$ can then be applied. With $F_\epsilon = \rho + \epsilon\sigma$, Eq. (5) shows $g_{\epsilon\, jk}$ has no vanishing eigenvalues because $\det g_\epsilon > 0$. The eigenvalues of $g_{\epsilon\, jk}$ are positive for $\epsilon = 0$, while no eigenvalue changes sign as $\epsilon$ grows to 1 because none vanishes. This line of reasoning also confirms invertibility of $g_{jk}$.

The comparative elegance of the result (5) is reminiscent of the analogous expressions for Randers space [2] and $b$ space [3]. In fact, Randers space is covered by two copies of the bipartite space (1) with opposite signs and with $s_{jk} = a_j a_k$, while the $b$ structure is a special case of Eq. (1) with $s_{jk} = b^2 r_{jk} - b_j b_k$. The result (5) for these cases is related via Theorem 2.3 of Ref. [27] to the metric determinant for general $(\alpha, \beta)$ spaces, which have Finsler structures of the form $F_{(\alpha,\beta)} = \alpha\phi(\beta/\alpha)$ for some $C^\infty$ positive function $\phi$, where $\alpha = \rho$ and $\beta$ is a 1-form on $TM\backslash S$. In this context, the Randers structure $F_a$ appears as an $(\alpha, \beta)$ structure with $\alpha = \rho$, $\beta = a \cdot y$, and $\phi = 1 + \beta/\alpha$. Also, Shen has observed [28] that the $b$ structure $F_b$ with constant norm $\|b\|$ can be viewed as an $(\alpha, \beta)$ structure with $\alpha = \rho$, $\beta = b \cdot y/\|b\|$, and $\phi = 1 \pm \|b\|\sqrt{1 - (\beta/\alpha)^2}$, with metric determinant given by Lemma 1.1.2 of Ref. [29]. Even for the more complicated $F_{ab}$ structure of $ab$ space [3], a relatively compact result exists for the metric determinant. Javaloyes and Sánchez have recently studied more general homogeneous functionals of Finsler structures and 1-forms [30], including the $(F_0, \beta)$ spaces generated as $\beta$-deformations of a Finsler structure $F_0$ [31] and the $(F_1, F_2)$ spaces generated by combining two Finsler structures $F_1$ and $F_2$. The $F_{ab}$ structure is a special case of an $(F_0, \beta)$ structure with $F_0 = F_b$, $\beta = a \cdot y$, and $\phi = 1 + \beta/F_0$, so the metric determinant is given by Proposition 4.24 of Ref. [30]. Modulo possible technical issues with the slit extension $S_1$, the bipartite structure discussed here takes the form of an $(F_1, F_2)$ space with $F_1 = \rho$, $F_2 = \sigma$, and $\phi = 1 + F_2/F_1$, although the result (5) appears unexpectedly simple given that $F_2$ is constructed from a bilinear form $s_{jk}$. Together with the existence of numerous other Finsler spaces arising from the motion of fermions in the SME [3], this simplicity suggests that further attractive Finsler geometries related to Lorentz violation in effective field theory remain to be discovered.

If the rank $m$ of $s$ is nonextremal, $0 < m < n$, then the image and kernel subspaces of $s$ are nontrivial. Since these spaces are orthogonal, we can uniquely project any vector $y$ in $TM_x$ into components parallel and perpendicular to the image subspace,

$$y^\parallel := \frac{1}{\varsigma} s y, \qquad y^\perp := y - y^\parallel. \tag{6}$$

Since $y^\parallel r y^\perp = 0$, the three vectors $y$, $y^\parallel$, and $y^\perp$ can be viewed as forming a right-angle triangle. Their norms satisfy the inequalities $\|y^\parallel\| \leqslant \|y\|$ and $\|y^\perp\| \leqslant \|y\|$, which are useful for several purposes. For example, for $s$ obeying Eq. (4) with $\varsigma < 1$, the bipartite structure $F$ is positive. This result can be viewed as a consequence of the inequality $\|y\| > \|y^\parallel\|$ for $y \neq y^\parallel$, which implies $\rho > \sqrt{ysy}/\sqrt{\varsigma}$, hence $\rho - \sqrt{ysy} > 0$, and thus $\rho + \sigma > 0$. As another example, we can apply the inequality $\|y^\parallel\| \leqslant \|y\|$ to show the sign of the function $S$ introduced in Eq. (5) matches that of $\sigma$, a result used above to prove strong convexity of $F$. For positive $\sigma$, $S$ is positive by inspection. Noting that $\sigma = \pm\sqrt{\varsigma}\|y^\parallel\|$, for negative $\sigma$ we can write $S = \varsigma\|y\| - \sqrt{\varsigma}\|y^\parallel\| \leqslant \sqrt{\varsigma}(\sqrt{\varsigma} - 1)\|y\| < 0$. The sign of $S/\sigma$ is therefore always positive, as claimed.

In terms of the projected vectors (6), the contribution $\sigma$ to the bipartite structure $F$ can be written in the form $\sigma = \pm\sqrt{\varsigma}\sqrt{yry^\parallel}$. However, the vectors $y^\parallel$ and $y^\perp$ play analogous roles in the triangle. This suggests the perpendicular component $y^\perp$ can be used to define a complementary bipartite structure $F^\perp$ given by

$$F^\perp := \rho + \sigma^\perp, \quad \sigma^\perp := \pm\sqrt{\varsigma}\sqrt{yry^\perp} = \pm\sqrt{\varsigma y^2 - ysy}, \tag{7}$$

where the sign choice can be independent of that adopted for $\sigma$. Up to a possible sign, the map $F \to F^\perp$ is thus implemented by the replacement

$$s \to s^\perp := \varsigma r - s, \tag{8}$$

which induces $\sigma \to \sigma^\perp$, $S \to S^\perp = \varsigma\rho + \sigma^\perp$, $\kappa_j \to \kappa^\perp{}_j = \rho_{y^j}/\rho - \sigma^\perp{}_{y^j}/\sigma^\perp$, etc. For example, using this replacement the corresponding Finsler metric $g^\perp{}_{jk}$, its determinant $\det(g^\perp{}_{jk})$, and the Cartan torsion $C^\perp{}_{jkl}$ can be obtained from Eqs. (2), (5), and (3), respectively. Note that a second iteration recovers $s$, $s \to \varsigma r - s \to s$, so the replacement (8) is a reflection. Also, in terms of the idempotent linear operator $\hat{s}^j{}_k$ the replacement gives $\hat{s} \to I - \hat{s}$, so in a suitable basis it amounts to the substitution $I_m \to I_{n-m}$.

With $0 < \varsigma < 1$ as before, the inequality $\|y\| > \|y^\perp\|$ for $y \neq y^\perp$ implies that $F^\perp$ is positive on $TM\backslash S$. Also, $F^\perp$ is positive homogeneous in $y^j$ of order one, and it is $C^\infty$ regular on the slit tangent bundle $TM\backslash S^\perp$, where $S^\perp = S_0 \cup {S_1}^\perp$ involves the perpendicular slit extension ${S_1}^\perp = \{y\colon s^j{}_k y^k = \varsigma y^j,\ \ y^j \neq 0\}$. Moreover, applying the standard argument [2] to the determinant $\det({g^\perp}_{jk})$ verifies that $F^\perp$ has strong convexity. These results imply that $F^\perp$ is a Finsler structure.

The above line of reasoning shows that bipartite Finsler structures obeying the condition (4) always appear in complementary pairs, $F$ and $F^\perp$. One example of such a pairing is provided by the Randers structure $F_a$ and the $b$ structure $F_b$ [3]. Another example involving $H$ space is presented below.

We remark in passing that both $F$ and $F^\perp$ can be expressed in terms of the Gram determinant or gramian, which for two vectors $y$, $z$ is $\mathrm{gram}(y,z) = y^2 z^2 - (y\cdot z)^2$. Noting that $\mathrm{gram}(y, sy) = \sigma^2 {\sigma^\perp}^2$, we find

$$F = \rho \pm \sqrt{\mathrm{gram}\big(y, sy/\sigma^\perp\big)}, \qquad F^\perp = \rho \pm \sqrt{\mathrm{gram}\big(y, sy/\sigma\big)}. \tag{9}$$

This generalizes the gramian expressions for $F_a$ and $F_b$ given in Ref. [3].

Using the determinant (5), we can calculate the mean Cartan torsion $I_j = (\ln(\det g))_{y^j}/2$ for $F$,

$$I_j = -\frac{1}{2}\left[(n+1)\frac{\sigma}{F} - (m-1)\frac{\varsigma\rho}{S}\right]\kappa_j. \tag{10}$$

Combining this with the Cartan torsion (3) yields the Matsumoto torsion

$$M_{jkl} = -\frac{1}{2}F\sum_{(jkl)}\kappa_j\left[\frac{m-1}{n+1}\frac{\varsigma\rho}{S}(\rho_{y^k y^l} + \sigma_{y^k y^l}) - \sigma_{y^k y^l}\right]. \tag{11}$$

The corresponding expressions ${I^\perp}_j$ and ${M^\perp}_{jkl}$ for the complementary bipartite structure $F^\perp$ can be obtained via the map (8). They take the same forms (10) and (11) with the substitutions $F \to F^\perp$, $\sigma \to \sigma^\perp$, $S \to S^\perp$, $\kappa_j \to {\kappa^\perp}_j$, and $m \to n - m$.

Except for special examples, notably the rank-1 cases, the Matsumoto torsions $M_{jkl}$ and ${M^\perp}_{jkl}$ are nonzero and so the Matsumoto–Hōjō theorem [32] shows that $F$ and $F^\perp$ typically differ from the Randers structure $F_a$ despite their apparent simplicity. Moreover, as we show explicitly below using $H$ space, only a subset of the bipartite $F$ and $F^\perp$ structures generate $b$ space. Interesting novel cases are therefore contained within Finsler structures built from $s_{jk}$ satisfying the condition (4). One intriguing open question in this context is identifying a new torsion that distinguishes $b$ space from other Finsler spaces, in analogy with the role of the Matsumoto torsion in distinguishing Randers space from other Finsler spaces. The simplicity of $b$ space, the complementary nature of $F_b$ to the Randers structure $F_a$, and the chirality relationship arising in the SME context between the pseudo-Finsler structures associated with $F_a$ and $F_b$ all are suggestive indications that such a torsion exists.

Since any $r$-parallel $b$ space is known to be Berwald [3], it is natural to ask whether a similar result holds for $r$-parallel bipartite spaces satisfying the condition (4). We can investigate this and obtain some related results by considering the geodesics associated with $F$, which obey

$$F\frac{d}{d\lambda}\left(\frac{1}{F}\frac{dx^j}{d\lambda}\right) + G^j = 0, \tag{12}$$

where the spray coefficients $G^j := g^{jm}\Gamma_{mkl}y^k y^l$ are defined in terms of the Christoffel symbol $\Gamma_{jkl}$ for $g_{jk}$. The first step towards obtaining the spray coefficients $G^j$ is to evaluate $G_j$ using $G_j = \Gamma_{jkl}y^k y^l$. We find

$$G_j = \rho F\tilde{\gamma}_{j\bullet\bullet} + \rho^2(\partial_\bullet\sigma - \sigma\tilde{\gamma}_{\bullet\bullet\bullet})\kappa_j + \frac{\rho^2 F}{\sigma}\hat{\gamma}_{j\bullet\bullet}, \tag{13}$$

where a lower index $m$ contracted with $r^{mk}\rho_{y^k}$ is denoted by a bullet $\bullet$, with contractions external to any derivatives that appear. The Christoffel symbol for the Riemann metric $r_{jk}$ is denoted $\tilde{\gamma}_{jkl}$, while that for $s_{jk}$ is denoted $\hat{\gamma}_{jkl}$. Note that some expressions involving $\hat{\gamma}_{jkl}$ can be more compactly expressed using the $r$-covariant derivative $\tilde{D}_j$ and the relationship

$$\hat{\gamma}_{jkl}|_{\partial\to\tilde{D}} := \frac{1}{2}(\tilde{D}_k s_{jl} + \tilde{D}_l s_{jk} - \tilde{D}_j s_{kl}) = \hat{\gamma}_{jkl} - s_{jm}\tilde{\gamma}^m{}_{kl}. \tag{14}$$

To find the spray coefficients $G^j$, we need the inverse bipartite metric $g^{jk}$. Since $g_{jk}$ is positive definite, the inverse metric exists. After some calculation, we find

$$g^{kl} = \frac{\rho}{F}\left(r^{kl} + \frac{{\sigma^\perp}^2\rho}{\sigma^2 S}\lambda^k\lambda^l - \frac{\rho}{S}s^{kl}\right), \tag{15}$$

where

$$\lambda_j := \frac{1}{\sigma^\perp}\left(s_{jk}y^k - \frac{\sigma S}{F}\rho_j\right). \tag{16}$$

For the complementary structure $F^\perp$, the inverse metric ${g^\perp}_{jk}$ is again obtained via the replacement (8). These results are similar in form to the expressions (22) and (23) of Ref. [3] for the inverse Finsler metric of $b$ space.

Using Eq. (15), a calculation shows that the bipartite spray coefficient $G^j$ can be written as

$$\begin{aligned} G^j = {} & \rho^2\tilde{\gamma}^j{}_{\bullet\bullet} + \frac{\rho^3}{S\sigma^3}\big[\sigma^3\hat{\gamma}^j{}_{\bullet\bullet} + \rho\sigma^2 {s^\perp}^{jk}\hat{\gamma}_{k\bullet\bullet} \\ & - \rho\sigma^\perp(\sigma^\perp\hat{\gamma}_{\circ\bullet\bullet} + \sigma\hat{\gamma}_{\bullet\bullet\bullet})\lambda^j\big]_{\partial\to\tilde{D}}, \end{aligned} \tag{17}$$

where an index $\circ$ represents a lower index $m$ contracted with $(s^\perp y)^m/\sigma^\perp$ externally to any derivatives. Note that the replacement (8) can be used to obtain the expression for the complementary spray coefficients ${G^\perp}^j$, which satisfy a geodesic equation for $F^\perp$ taking the form (12).

The result (17) reveals that $G^j$ contains the standard term $\tilde{\gamma}_{jkl}y^j y^k$ together with a linear combination of terms, each of which involves the Riemann covariant derivative acting on $s_{jk}$. It follows from (17) that if the bipartite form $s_{jk}$ is $r$-parallel, $\tilde{D}_l s_{jk} = 0$, then the spray coefficients $G^j$ reduce to the usual Riemann case and the trajectories satisfy the usual Riemann geodesic equation. In this situation the spray coefficients are quadratic in $y^j$, so the third $y^j$ derivative of $G^j$ is zero, and therefore the Berwald h-v curvature ${}^B P_k{}^j{}_{lm} := -F(G^j)_{y^k y^l y^m}/2$ vanishes. We can conclude that any $r$-parallel bipartite space satisfying the condition (4) is necessarily Berwald. The same result follows for the bipartite space with complementary structure $F^\perp$. It would be of interest to investigate the validity of the converse hypothesis that any bipartite Berwald space obeying the condition (4) is $r$-parallel. In any case, the result established above is consistent with the conjecture that any SME-based Riemann–Finsler space is Berwald iff it has $r$-parallel coefficients for Lorentz violation [3]. Since the presence of nonzero $r$-parallel $s^j{}_k$ leaves geodesics unaffected, the result also is indicative of the existence of a variable transformation or redefinition that would eliminate $s^j{}_k$ in this limit, just as certain unphysical coefficients can be eliminated in the SME [4,33,13,35–37]. Investigation of these two open conjectures is likely to lead to additional mathematical and physical insights.

**Table 1**
Isomorphisms between Riemann, Randers, $b$, $H$, $H^\perp$, and bipartite spaces.

| $n$ | $m=0$ | $m=1$ | $m=2$ | $m=3$ | $m=4$ | $m=5$ | $m=6$ | $\cdots$ | $m=n-2$ | $m=n-1$ | $m=n$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $r_1$ | $r_1^\perp$ | | | | | | | | | |
| 2 | $r_2$ | $a_2=b_2$ | $r_2^\perp$ | | | | | | | | |
| 3 | $r_3$ | $a_3=H_{3,2}^\perp$ | $b_3=H_{3,2}$ | $r_3^\perp$ | | | | | | | |
| 4 | $r_4$ | $a_4$ | $H_{4,2}=H_{4,2}^\perp$ | $b_4$ | $r_4^\perp$ | | | | | | |
| 5 | $r_5$ | $a_5=H_{5,4}^\perp$ | $H_{5,2}$ | $H_{5,2}^\perp$ | $b_5=H_{5,4}$ | $r_5^\perp$ | | | | | |
| 6 | $r_6$ | $a_6$ | $H_{6,2}=H_{6,4}^\perp$ | $s_{6,3}$ | $H_{6,4}=H_{6,2}^\perp$ | $b_6$ | $r_6^\perp$ | | | | |
| $\vdots$ | | | | | | | | $\vdots$ | | | |
| odd | $r_n$ | $a_n=H_{n,n-1}^\perp$ | $H_{n,2}$ | $H_{n,n-3}^\perp$ | $H_{n,4}$ | $H_{n,n-5}^\perp$ | $H_{n,6}$ | $\cdots$ | $H_{n,2}^\perp$ | $b_n=H_{n,n-1}$ | $r_n^\perp$ |
| even | $r_n$ | $a_n$ | $H_{n,2}=H_{n,n-2}^\perp$ | $s_{n,3}$ | $H_{n,4}=H_{n,n-4}^\perp$ | $s_{n,5}$ | $H_{n,6}=H_{n,n-6}^\perp$ | $\cdots$ | $H_{n,n-2}=H_{n,2}^\perp$ | $b_n$ | $r_n^\perp$ |

The $y$-derivative $p_j := F_{y^j}$ of a Finsler structure plays an important role in both mathematics and physics. Mathematically, $p_j$ defines the Hilbert form $F_{y^j}dx^j$. Physically, the corresponding quantity for a pseudo-Finsler structure is the canonical momentum. The $y$-derivative $p_j$ determines an algebraic variety $\mathcal{R}(p)$, which is the dispersion relation governing the geodesic motion. For the bipartite structure (1), $p_j$ takes the form $p_j = r_{jk}y^k/\rho + s_{jk}y^k/\sigma$. Restricting attention to $F$ obeying the condition (4), we find the dispersion relation can be written as

$$\left(p^2 - 1 + \varsigma\right)^2 - 4psp = 0. \tag{18}$$

The corresponding result for the complementary structure $F^\perp$ is obtained by the replacement (8). For example, the dispersion relation for the Randers structure $F_a$ is given by Eq. (18) with $s_{jk} = a_j a_k$, while that for the $b$ structure $F_b$ follows when $s_{jk} = b^2 r_{jk} - b_j b_k$. These expressions are the Finsler versions of the pseudo-Finsler dispersion relations derived for the motion of a classical fermion in the presence of nonzero SME coefficients $a_\mu$ and $b_\mu$ in $(3+1)$-dimensional spacetime [33], the effects of which have been sought in numerous experiments [34]. General descriptions of Lorentz-violating dispersion relations can be found in Refs. [38–40,37].

Another interesting SME coefficient is the 2-form $H_{\mu\nu}$, which arises naturally in some models with spontaneous Lorentz breaking [41] and for which the dispersion relation is also known [42]. Physical effects from $H_{\mu\nu}$ have been studied in the electron sector using a torsion pendulum [43], in the neutron sector with a He–Xe comagnetometer [44], in the muon sector in a storage ring [45], and in the neutrino sector using neutrino oscillations [46]. For the generic case the form of the associated pseudo-Finsler structure is presently unknown, but the special case with vanishing quadratic invariant $Y = \epsilon^{\alpha\beta\gamma\delta}H_{\alpha\beta}H_{\gamma\delta}/8$ yields a calculable example [13]. Associated with these pseudo-Riemann–Finsler spaces is a Finsler geometry, $H$ space, that involves a 2-form $H_{jk}$ [3].

Here, we consider a bipartite limit of the $H$ geometry obtained via a suitable constraint on the linear operator $H^j{}_k = r^{jl}H_{lk}$. The antisymmetry of $H_{jk}$ implies $H^j{}_k = -H_k{}^j$, so $H^j{}_k$ has even rank. In odd dimensions it therefore has at least one zero eigenvalue, while the total number of zero eigenvalues is odd in odd dimensions and is even in even dimensions. The quadratic product $(H^2)^j{}_k = H^j{}_l H^l{}_k$ obeys $(H^2)^j{}_k = (H^2)_k{}^j$, and all its nonzero eigenvalues are negative. We focus attention on the restricted class of $H^j{}_k$ for which $(H^2)^j{}_k$ has only a single nonzero eigenvalue $-\eta$, so that

$$H^4 = -\eta H^2. \tag{19}$$

Since the condition (19) is of the form (4), we may define a bipartite $H$ space by identifying $s = -H^2$, $\varsigma = \eta$. The associated Finsler structure $F_H$ and its complementary structure $F_H^\perp$ are

$$F_H = \rho \pm \sqrt{-yH^2y}, \qquad F_H^\perp = \rho \pm \sqrt{\eta y^2 + yH^2y}, \tag{20}$$

where the sign choices in the two expressions can be independent. In terms of the gramian, we can write

$$F_H = \rho \pm \sqrt{\mathrm{gram}\left(y, -H^2y/\sigma^\perp\right)} = \rho \pm \sqrt{\mathrm{gram}(y, Hy/\rho)}, \tag{21}$$

where the first expression is of the type (9) and the second exploits the antisymmetric nature of $H_{jk}$.

The basic properties of this restricted $H$ space follow by applying the results for $s_{jk}$ satisfying the condition (4). The Finsler metric for $F_H$ takes the form (2) with $s_{jk} = -(H^2)_{jk}$, while the metric determinant is given by Eq. (5) and its inverse by Eq. (15). The Cartan torsion and its mean, the Matsumoto torsion, the spray coefficients, and the dispersion relation are all given by substitution into formulae presented above. The analogous results for the complementary structure $F_H^\perp$ can be found by the replacement $(H^2)_{jk} \to \eta r_{jk} + (H^2)_{jk}$. Note that one key difference between the restricted $H$ space and the bipartite space obeying the condition (4) is that the rank $m$ is necessarily even for $H$ space. Note also that the complementary bipartite structure $F_H^\perp$ is the $n$-dimensional Finsler analogue of the $(3+1)$-dimensional pseudo-Finsler structure given in Eq. (15) of Ref. [13], while the dispersion relation (18) for $F_H^\perp$ is the $n$-dimensional Finsler analogue of the $(3+1)$-dimensional dispersion relation for $Y = 0$.

As seen above, Riemann space, Randers space, $b$ space, and the two restricted $H$ spaces are all examples of bipartite spaces obeying the condition (4). Any such bipartite space is fixed by specifying the dimension $n$ of the configuration space, the rank $m$ of $s^j{}_k$, and the eigenvalue $\varsigma$. This implies certain spaces are isomorphic. For example, Randers space and $b$ space are isomorphic in two dimensions when $b_j = a_j$ because both have $n = 2$, $m = 1$, and $\varsigma = a^2$. To express these isomorphisms compactly, it is convenient to introduce notation for the various spaces. For dimension $n$ and rank $m$, let $s_{n,m}$ be the bipartite space obeying the condition (4). If $m = 0$, then it suffices to indicate $n$ and the space is Riemann, denoted $r_n$. The case $m = n$ yields the complementary Riemann space with scaled metric, written $r_n^\perp$. The rank $m$ is always 1 for the Randers spaces $a_n$, while the rank $n-1$ is fixed by the dimension for the $b$ spaces $b_n$. The restricted $H$ space in $n$ dimensions with $(H^2)^j{}_k$ of rank $m$ is denoted $H_{n,m}$, and the complementary space is written $H_{n,m}^\perp$.

Using these conventions and assuming a definite value of $\varsigma$, Table 1 summarizes the isomorphisms between the various cases. Each cell in the table represents an $s_{n,m}$ space with specified $n$ and $m$. Note that cells with $m > n$ are meaningless and are left blank. Most of the $s_{n,m}$ spaces can be identified with one or more of the other spaces, so we use $s_{n,m}$ only where no other notation applies. Only for even $n$ with certain odd $m$ do $s_{n,m}$ spaces exist that are distinct from the $a_n$, $b_n$, $H_{n,m}$, and $H_{n,m}^\perp$ spaces. The first three occurrences of this are $s_{6,3}$ in six dimensions and $s_{8,3}$ and $s_{8,5}$ in eight dimensions. For ranks $m = 0$ and $m = n$, Riemann spaces are obtained, and these have no isomorphisms with other bipartite spaces because the Cartan torsion (3) vanishes. The rank-one Randers spaces $a_n$ in odd dimensions are isomorphic to the complementary $H$ spaces $H^{\perp}_{n,n-1}$, while in even dimensions they are unique except for the isomorphism with $b$ space for $n=2$. Analogously, the rank-$(n-1)$ spaces $b_n$ in odd dimensions are isomorphic to $H_{n,n-1}$, while in even dimensions they are unique except for $b_2 = a_2$. For other ranks, the $s_{n,m}$ spaces in odd dimensions generate an alternating series of restricted $H$ spaces and their complements. Also, each restricted $H$ space with even rank and dimension is isomorphic to a complementary $H$ space, $H_{n,m} = H^{\perp}_{n,n-m}$. The general cases for odd and even dimensions are listed in the last two rows of the table.

As a final remark, we note that the comparatively simple Finsler structure associated with bipartite geometries obeying the condition (4) and the variety of isomorphisms displayed in Table 1 together suggest the potential for interesting physical applications of Eq. (1) in addition to the pseudo-Riemann–Finsler applications to the SME mentioned above. For example, Shen [47,48] has demonstrated that Randers geodesics correspond to solutions of the Zermelo navigation problem of navigation control in an external wind related to the coefficient $a_j$. This result provides a direct physical application of the spaces with $m=1$ listed in the third column of Table 1. Finding analogous physical interpretations for the other entries in the table is an intriguing open challenge.

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